N-level quantum systems with SU(2) dynamic symmetry

Foek T. Hioe
Saint John Fisher University, fhioe@sjf.edu

Publication Information

Hioe, Foek T. (1987). "N-level quantum systems with SU(2) dynamic symmetry." *Journal of the Optical Society of America B* 4.8, 1327-1332.

N-level quantum systems with SU(2) dynamic symmetry

F. T. Hioe

Department of Physics, St. John Fisher College, Rochester, New York 14618

Received February 26, 1987; accepted April 28, 1987

We present an analytic solution of the coherent evolution of a laser-driven N-level quantum system that possesses an SU(2) dynamic symmetry.

In this paper we present an analytic solution of the coherent evolution of a laser-driven N-level quantum system that possesses an SU(2) dynamic symmetry. This SU(2) model, as we shall call it, includes the Cook–Shore model[1] as a special case. We shall give, in particular, the conditions for getting complete population transfer, complete population return, and complete population depletion in simple and explicit expressions.

The coherent dynamics of multiple-laser excitation of atoms or molecules involving N levels or states is described by the time-dependent Schrödinger equation for the probability amplitude $C_n(t)$, $n = 1, 2, \ldots, N$, for the nth level that can be written as (in units of $\hbar = 1$)

$$i\frac{\mathrm{d}}{\mathrm{d}t} C_n(t) = \sum_{n'=1}^{N} H_{nn'}(t)C_{n'}(t)$$

or

$$i\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{C}(t) = \hat{H}(t)\mathbf{C}(t) \tag{1}$$

in the matrix form, where $H_{nn'}(t)$ are the matrix elements of the Hamiltonian operator $\hat{H}(t)$. For many problems in which a rotating-wave approximation is used and in which the dipole transition moments link the levels only stepwise, $1 \leftrightarrow 2, 2 \leftrightarrow 3, \ldots, N-1 \leftrightarrow N$, the matrix elements of $\hat{H}(t)$ can be written as[1–4]

$$H_{nn}(t) \equiv \Delta_n(t), \tag{2}$$

$$H_{n,n+1}(t) = H^*_{n+1,n}(t) \equiv -\tfrac{1}{2}\Omega_n(t), \tag{3}$$

and

$$H_{nn'}(t) = 0 \qquad \text{otherwise,}$$

where $\Delta_n(t)$ is the cumulative detuning of $n-1$ successive lasers from the corresponding sum of $n-1$-level frequencies and $\Omega_n(t)$ is the Rabi frequency for step n given in terms of the dipole transition moment d_n linking levels n and $n+1$ and the possibly complex-valued and time-dependent electric-field amplitude $\mathcal{E}_n(t)$, whose carrier frequency matches the frequency for the $n \to n+1$ transition, by

$$\Omega_n(t) = 2d_n\mathcal{E}_n(t). \tag{4}$$

Just as the Cook–Shore model[1] was constructed to be mathematically analogous to a spin j system in a constant magnetic field, our SU(2) model is constructed to be analogous to a spin j system in a time-dependent magnetic field. More specifically, we assume that the Hamiltonian $\hat{H}(t)$ of our N-level SU(2) model lies entirely in the subspace spanned by the generators of the O(3) subgroup of the N^2-1 generators of the SU(N) algebra, i.e., we assume that $\hat{H}(t)$ can be written as

$$\hat{H}(t) = c_1(t)\hat{J}_x + c_2(t)\hat{J}_y + c_3(t)\hat{J}_z + d(t), \tag{5}$$

where $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ are the angular-momentum operators of spin $j = \tfrac{1}{2}(N-1)$ and $c_1(t)$, $c_2(t)$, $c_3(t)$, and $d(t)$ are arbitrary functions of time. When the Hamiltonian operator of the system can be written as in Eq. (5), the system is said to possess an SU(2) dynamic symmetry. One of the important consequences of this symmetry is that the N^2-dimensional dynamic space in which the N^2 elements of the density matrix $\hat{\rho}(t)$ evolve can be shown[5,6] to be decomposable into N-independent subspaces of dimensions $1, 3, 5, \ldots, 2N-1$.

Following the usual notation in the angular-momentum algebra, the index m is used, it takes up the values $-j$, $-j+1, -j+2, \ldots, j$, and it is related to the index n that takes up the values $1, 2, \ldots, N$ by

$$m = n - j - 1, \qquad N = 2j + 1. \tag{6}$$

Using the familiar irreducible matrix representation for the angular-momentum operators for which $\hat{J}_z$ is diagonal,[7] Eq. (5) implies that $\hat{H}(t)$ is tridiagonal, with diagonal elements

$$\begin{aligned} H_{nn}(t) = \Delta_n(t) &= mc_3(t) + d(t) \\ &= nc_3(t) + [d(t) - \tfrac{1}{2}c_3(t)(N+1)] \end{aligned} \tag{6a}$$

and off-diagonal elements

$$\begin{aligned} H_{n+1,n}(t) = H^*_{n,n+1}(t) &= \tfrac{1}{2}[c_1(t) + ic_2(t)] \\ &\quad \times [j(j+1) - m(m+1)]^{1/2} \\ &= \tfrac{1}{2}[c_1(t) + ic_2(t)]\sqrt{n(N-n)}. \end{aligned} \tag{6b}$$

Since the presence of $d(t)$ in the diagonal elements changes only the result for the probability amplitudes by the same phase factor, we ignore its presence and write

$$H_{nn}(t) = \Delta_n(t) = -m\Delta_0(t), \qquad m = -j, -j+1, \ldots, j \tag{7a}$$

and

0740-3224/87/081327-06$02.00

$$H_{n+1,n}(t) = H^*_{n,n+1}(t) = -\tfrac{1}{2}\Omega_n(t) = -\tfrac{1}{2}\sqrt{n(N-n)}\,\Omega_0(t), \quad n = 1, 2, \ldots, N, \tag{7b}$$

where $\Delta_0(t)$ and $\Omega_0(t)$ are arbitrary functions of time and are the two principal independent (generally complex and time-dependent) parameters of our SU(2) model.

We shall write the probability amplitudes in Eqs. (1) as $C_m{}^{(j)}(t)$ and label their subscripts by $m = -j, -j+1, \ldots, j$. The formal solution of Eqs. (1) is

$$\mathbf{C}(t) = \left[\exp - i \int \hat{H}(t)\mathrm{d}t\right]\mathbf{C}(0).$$

If $\hat{H}(t)$ is given by Eq. (5), as our SU(2) model assumes, then the vectors $\mathbf{C}(t)$ are transformed among themselves for different values of t by the transformations of the SU(2) group. For the two-level system $N = 2$ or $j = \tfrac{1}{2}$, Eqs. (1) become

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} C_{-1/2}{}^{(1/2)}(t) \\ C_{1/2}{}^{(1/2)}(t)\end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}\Delta_0(t) & -\tfrac{1}{2}\Omega_0(t) \\ -\tfrac{1}{2}\Omega^*{}_0(t) & -\tfrac{1}{2}\Delta_0(t)\end{bmatrix}\begin{bmatrix} C_{-1/2}{}^{(1/2)}(t) \\ C_{1/2}{}^{(1/2)}(t)\end{bmatrix}, \tag{8}$$

and let us write the solution for the C's at time t in terms of their values at the initial time $t = 0$ as

$$\begin{bmatrix} C_{-1/2}{}^{(1/2)}(t) \\ C_{1/2}{}^{(1/2)}(t)\end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ -b^*(t) & a^*(t)\end{bmatrix}\begin{bmatrix} C_{-1/2}{}^{(1/2)}(0) \\ C_{1/2}{}^{(1/2)}(0)\end{bmatrix}, \tag{9}$$

where $a(t)$ and $b(t)$ satisfy

$$|a(t)|^2 + |b(t)|^2 = 1 \tag{10}$$

and will be referred to as the fundamental solution of the two-level system [Eq. (8)] in the sense that $C_{-1/2}{}^{(1/2)}(t) = a(t)$ [if the initial conditions are $|C_{-1/2}{}^{(1/2)}(0)| = 1$ and $C_{1/2}{}^{(1/2)}(0) = 0$] and that $C_{-1/2}{}^{(1/2)}(t) = b(t)$ [if the initial conditions are $C_{-1/2}{}^{(1/2)}(0) = 0$ and $|C_{1/2}{}^{(1/2)}(0)| = 1$]. Equation (9) also expresses the fact that $\mathbf{C}^{(1/2)}(t)$ is transformed for different values of t by the two-dimensional representation of the unitary group $D^{(1/2)}(a, b)$, which is the matrix on the right-hand side of Eq. (9). It follows that, if $a(t)$ and $b(t)$ can be found from Eq. (8), the $2j + 1$-dimensional representation of the unitary group $D^{(j)}(a, b)$ immediately provides us with the solution of our $N(=2j + 1)$-level SU(2) model in that the solution of Eqs. (1) in which $\hat{H}(t)$ is given by Eqs. (7) is given immediately by

$$\mathbf{C}(t) = D^{(j)}(a, b)\mathbf{C}(0) \tag{11}$$

or

$$C_m{}^{(j)}(t) = \sum_{m'=-j}^{j} D^{(j)}_{mm'}[a(t), b(t)]C_{m'}{}^{(j)}(0), \quad m = -j, -j+1, \ldots, j. \tag{12}$$

The matrix elements $D^{(j)}_{mm'}(a, b)$ are given in many standard texts.[8] The following expression:

$$D^{(j)}_{mm'}(a, b) = \sum_{\mu} \frac{[(j-m)!(j+m)!(j-m')!(j+m')!]^{1/2}}{p!q!r!s!} \times a^p a^{*q} b^r (-b^*)^s, \tag{13}$$

where

(i) $p = j - m - \mu, \quad q = j + m' - \mu, \quad r = \mu, \quad s = m - m' + \mu$

or

(ii) $p = -m - m' + \mu, \quad q = \mu, \quad r = j + m' - \mu, \quad s = j + m - \mu$

or

(iii) $p = \mu, \quad q = m + m' + \mu, \quad r = j - m - \mu, \quad s = j - m' - \mu$

and where $\mu = 0, 1, 2, \ldots$ can all be used and will give the same result; however, each has its own advantage when one wants to see some specially simple expressions for $D^{(j)}_{mm'}$, ones that arise, for example, when m or m' is equal to $-j$ or j.

Specifically, we have

$$D^{(j)}_{m,-j}(a, b) = \left[\frac{(2j)!}{(j-m)!(j+m)!}\right]^{1/2} a^{j-m}(-b^*)^{j+m}, \tag{14a}$$

$$D^{(j)}_{-j,m}(a, b) = \left[\frac{(2j)!}{(j-m)!(j+m)!}\right]^{1/2} a^{j-m} b^{j+m}, \tag{14b}$$

$$D^{(j)}_{m,j}(a, b) = \left[\frac{(2j)!}{(j-m)!(j+m)!}\right]^{1/2} a^{*j+m} b^{j-m}, \tag{14c}$$

$$D^{(j)}_{j,m}(a, b) = \left[\frac{(2j)!}{(j-m)!(j+m)!}\right]^{1/2} a^{*j+m}(-b^*)^{j-m}. \tag{14d}$$

Also, when j = integer and $m = m' = 0$, we have

$$D^{(j)}_{00}(a, b) = \sum_{m=0}^{i} (-1)^m \binom{j}{m}^2 |a|^{2(j-m)}|b|^{2m} = P_j(x), \tag{15}$$

where $x = |a|^2 - |b|^2$ and $P_n(x)$ is the Legendre polynomial of order n. The last equality in Eq. (15) can be seen from the following expression for the Jacobi polynomial:

$$P_n{}^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m}\binom{n+\beta}{m}\left(\frac{1+x}{2}\right)^{n-m}\left(\frac{1-x}{2}\right)^m$$

and from the relation that

$$P_n{}^{(0,0)}(x) = P_n(x).$$

The matrices $D^{(j)}$ for $j = \tfrac{1}{2}(N = 2)$, $1(N = 3)$, $\tfrac{3}{2}(N = 4)$, and $2(N = 5)$ are the following:

$$j=\frac{1}{2},\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix};\qquad j=1,\begin{bmatrix} a^2 & \sqrt{2}ab & b^2 \\ -\sqrt{2}ab^* & |a|^2-|b|^2 & \sqrt{2}a^*b \\ -b^{*2} & -\sqrt{2}a^*b^* & a^{*2} \end{bmatrix};$$

$$j=\frac{3}{2},\begin{bmatrix} a^3 & \sqrt{3}a^2b & \sqrt{3}ab^2 & b^3 \\ -\sqrt{3}a^2b^* & a(|a|^2-2|b|^2) & b(-|b|^2+2|a|^2) & \sqrt{3}a^*b^2 \\ \sqrt{3}ab^{*2} & b^*(-2|a|^2+|b|^2) & a^*(|a|^2-2|b|^2) & \sqrt{3}a^{*2}b \\ -b^{*3} & \sqrt{3}a^*b^{*2} & -\sqrt{3}a^{*2}b^* & a^{*3} \end{bmatrix};$$

$$j=2,\begin{bmatrix} a^4 & 2a^3b & \sqrt{6}a^2b^2 & 2ab^3 & b^4 \\ -2a^3b^* & a^2(|a|^2-3|b|^2) & \sqrt{6}ab(|a|^2-|b|^2) & b^2(3|a|^2-|b|^2) & 2a^*b^3 \\ \sqrt{6}a^2b^{*2} & \sqrt{6}ab^*(-|a|^2+|b|^2) & |a|^4-4|a|^2|b|^2+|b|^4 & \sqrt{6}a^*b(|a|^2-|b|^2) & \sqrt{6}a^{*2}b^2 \\ -2ab^{*3} & b^*(3|a|^2-|b|^2) & \sqrt{6}a^*b^*(-|a|^2+|b|^2) & a^{*2}(|a|^2-3|b|^2) & 2a^{*3}b \\ b^{*4} & -2a^*b^{*3} & \sqrt{6}a^{*2}b^{*2} & -2a^{*3}b^* & a^{*4} \end{bmatrix}. \quad (16)$$

If level 1 (labeled $m = -j$) is initially occupied with probability one, i.e., if $|C_{-j}^{(j)}(0)| = 1$ and $C_{-j+1}^{(j)}(0) = C_{-j+2}^{(j)}(0) = \ldots = C_j^{(j)}(0) = 0$, then the occupation probability of level n (labeled $m = n - j - 1$) at time t is given simply by

$$|C_m^{(j)}(t)|^2 = \binom{2j}{j-m}|a(t)|^{2(j-m)}|b(t)|^{2(j+m)},\qquad m=-j,-j+1,\ldots,j, \quad (17)$$

as can be easily seen from Eqs. (12) and (14).

Equations (12) and (17) for the specific initial condition express the solution of our N-level SU(2) model remarkably simply in terms of the fundamental solution $a(t)$ and $b(t)$ of the two-level system [Eq. (8)] for which a large number of analytic solutions are available for a variety of time-dependent Rabi frequency $\Omega_0(t)$ and detuning function $\Delta_0(t)$. Thus, for example, from the matrices $D^{(j)}$ given explicitly for $N = 2j + 1 = 3$, 4, and 5 in Eq. (16), the level population distributions at time t are given, in terms of the solution $A(t) \equiv |a(t)|^2$ and $B(t) \equiv |b(t)|^2$ of the two-level system [Eq. (8)] by simple expressions shown in Table 1 for different initial level population distributions. The results corresponding to the inverted initial level population distributions [such as (0, 1, 0, 0) and (0, 0, 1, 0)] are related by a simple interchange of A and B and are therefore not separately shown in Table 1. We note that since the transformation [Eq. (11)] is unitary, $\sum_{m=-j}^{j}|C_m^{(j)}(t)|^2$ is invariant; hence our solution [Eq. (12)] is normalized at any time t if $\sum_{m=-j}^{j}|C_m^{(j)}(0)|^2$ is normalized.

It is remarkable that our solution is not only more general but is also much simpler in form than the solution given by Cook and Shore,[1] whose method and formula [Eq. (18) of Ref. 1] cannot be easily extended to the case involving time-dependent $\Omega_0(t)$ and $\Delta_0(t)$ for which our solution applies. The use of $D^{(j)}(a, b)$, in which we identified $a(t)$ and $b(t)$ with the fundamental time-dependent solution of Eq. (8), instead of the use of the more physical $D^{(j)}(\alpha, \beta, \gamma)$, in which α, β, γ were associated with the Euler angles[9] chosen appropriately to diagonalize the Hamiltonian operator,[1] is seen to have led us to the successful generalization and simplification of the Cook–Shore result.

Table 1. Level Populations at Time t for N = 3, 4, 5 of Our SU(2) Model in Terms of the Solution $A \equiv |a(t)|^2$ and $B \equiv |b(t)|^2$ of Eq. (8)

	Initial	At Time t	Initial	At Time t	Initial	At Time t
$N = 3$						
	1	A^2	0	$2AB$		
	0	$2AB$	1	$(A-B)^2$		
	0	B^2	0	$2AB$		
$N = 4$						
	1	A^3	0	$3A^2B$		
	0	$3A^2B$	1	$A(A-2B)^2$		
	0	$3AB^2$	0	$B(-2A+B)^2$		
	0	B^3	0	$3AB^2$		
$N = 5$						
	1	A^4	0	$4A^3B$	0	$6A^2B^2$
	0	$4A^3B$	1	$A^2(A-3B)^2$	0	$6AB(A-B)^2$
	0	$6A^2B^2$	0	$6AB(-A+B)^2$	1	$(A^2-4AB+B^2)^2$
	0	$4AB^3$	0	$B^2(3A-B)^2$	0	$6AB(-A+B)^2$
	0	B^4	0	$4AB^3$	0	$6A^2B^2$

It may be remarked that, if we reduce the N coupled equations in Eqs. (1) into a single Nth-order differential equation for the first component $C_{-j}^{(j)}(t)$ [or the last component $C_j^{(j)}(t)$] by eliminating the other components $C_m^{(j)}(t)$, $m = -j+1, -j+2, \ldots, j$ from the coupled equations, then this Nth-order differential equation for $C_{-j}^{(j)}(t)$ is related to the second-order differential equation for $C_{-1/2}^{(1/2)}(t)$ obtained by eliminating $C_{1/2}^{(1/2)}(t)$ from Eq. (8) by Appell's theorem,[10] as can be seen by noting the simple feature of the first row (or the last row) of the matrix $D^{(j)}$ given by Eq. (14b) [or Eq. (14d)] and more specifically by Eqs. (16) and writing out $C_{-j}^{(j)}(t)$ in terms of $a(t)$ and $b(t)$.

From Eq. (17), it follows that the condition for complete transfer of population from level 1 ($m = -j$) to level $N(m = j)$ for our N-level SU(2) model coincides precisely with the condition for complete transfer of population from level 1 to level 2 in the two-level system [Eq. (8)] and that the condition for complete return of population from level 1 to level 1 for the N-level system also coincides with the condition for complete return of population from level 1 to level 1 in the two-level system [Eq. (8)].

For integral values of j or odd values of N ($N = 3, 5, 7, \ldots$), if the middle level labeled by $m = 0$ or $n = j + 1 = \frac{1}{2}(N+1)$ is initially occupied with probability one, i.e., if $|C_0^{(j)}(0)| = 1$, $C_m^{(j)}(0) = 0$ for $m \neq 0$, then it follows from Eq. (15) that the condition for that level to be completely depleted of population at time t is that

$$\sum_{m=0}^{j} (-1)^m \binom{j}{m}^2 |a(t)|^{2(j-m)} |b(t)|^{2m} = 0 \tag{18}$$

or

$$P_j(x) = 0, \tag{19a}$$

where

$$x = |a(t)|^2 - |b(t)|^2. \tag{19b}$$

Using Eq. (18) or (19), the values of $|a(t)|^2$ at time t, at which time the middle level of an N-level system will be completely depleted of population, are given in Table 2 for various values of N. The corresponding values of $|b(t)|^2$ are, of course, given by $1 - |a(t)|^2$. Since $P_j(x)$ is an odd function of x when j is odd, therefore $x = 0$ or $|a(t)|^2 = \frac{1}{2}$ is always a solution of Eqs. (19) for complete population depletion when $N = 4n - 1, n = 1, 2, 3, \ldots$, but $|a(t)|^2 = \frac{1}{2}$ is *never* a solution of Eqs. (19) for complete population depletion when $N = 4n + 1$, a rather curiously unexpected result. It is useful to note that all the roots of Legendre polynomials[11] are real, simple, and between -1 and 1 and that, between two consecutive zeros of $P_n(x)$, there is exactly one zero of $P_{n+1}(x)$ and at least one zero of $P_m(x)$ for each $m > n$. The number of possible solutions for complete population depletion for an odd $N = 2j + 1$ level SU(2) model is exactly equal to j.

When the following analytic expressions for $a(t)$ and $b(t)$ for the solution of Eq. (8) are substituted into Eq. (12), Eq. (17), or Table 1, they can be shown to reproduce the previously known results in simpler forms:

(1) *The Cook–Shore Result.*[1] Here we have Δ_0 = constant and Ω_0 = constant in Eqs. (7). The solution of Eq. (8) is

Table 2. The Values of $|a(t)|^2$ for which the Middle Level of a SU(2) Model N-Level System Will Have Zero Population at Time t

N	$\|a(t)\|^2$
3	½
5	0.789, 0.211
7	½, 0.887, 0.113
9	0.931, 0.670, 0.330, 0.0694
11	½, 0.953, 0.769, 0.231, 0.0469
$2j+1$	$\frac{1}{2} + \frac{1}{2}x_n, n = 1, 2, \ldots, j^a$

^{a} $x_n, n = 1, 2, \ldots, j$ denote the roots of $P_j(x) = 0$.

$$a = \cos \omega t - i \frac{\Delta_0}{2\omega} \sin \omega t, \qquad b = i \frac{\Omega_0}{2\omega} \sin \omega t \tag{20}$$

or

$$|a|^2 = 1 - \frac{|\Omega_0|^2}{4\omega^2} \sin^2 \omega t, \quad |b|^2 = \frac{|\Omega_0|^2}{4\omega^2} \sin^2 \omega t, \tag{21}$$

where

$$\omega = \tfrac{1}{2}\sqrt{|\Omega_0|^2 + \Delta_0^2}. \tag{22}$$

The case in which $\Delta_0(t)$ = constant $\times f(t)$ and $\Omega_0(t)$ = constant $\times f(t)$ can be reduced to the case above by changing the time scale of the problem from t to τ, where $d\tau = f(t)dt$.

Assuming that the initial occupation probabilities of levels 1 and 2 are 1 and 0, respectively, complete transfer of population requires that $\Delta_0 = 0$ and $\sin \omega t = n\pi, n = 1, 2, \ldots$, and complete return of population from level 1 to level 1 requires that $\sin \omega t = 2n\pi$. For the N-level system, the same conditions translate into those for complete transfer of population from level 1 to level N and those for complete return of population from level 1 to level 1, respectively. For the case when N is odd, assuming that the middle level is initially occupied with probability 1, the condition for that level to be completely depleted of population at time t can be obtained from Eq. (21) and Table 2. While Cook and Shore[1] did give some specific numerical results for the resonant case, the precise conditions for complete transfer, complete return, and complete depletion of population that we have just stated above were not given by them explicitly.

(2) *Adiabatic Following.*[12,13] Here $\Delta_0(t)$ and $\Omega_0(t)$ in Eqs. (7) are real arbitrary functions of time, except that $|\Delta_0(0)| \gg |\Omega_0(0)|$ is assumed. The solution of Eq. (8), subject to the adiabatic following condition, is:

$$a = \left(\tfrac{1}{2}\left\{1 + \frac{\Delta_0(t)}{[\Omega_0(t)^2 + \Delta_0(t)^2]^{1/2}}\right\}\right)^{1/2} e^{i\theta(t)}, \tag{23a}$$

$$b = \left(\tfrac{1}{2}\left\{1 - \frac{\Delta_0(t)}{[\Omega_0(t)^2 + \Delta_0(t)^2]^{1/2}}\right\}\right)^{1/2} e^{i\theta(t)}, \tag{23b}$$

where $\theta(t)$ is an arbitrary phase factor. Substitutions of Eqs. (23) into Eq. (12) can be verified to give the same result[14] as Eqs. (17) of Ref. 12.

Among the new analytic results for our N-level SU(2) model are those that can be deduced from the following recently derived analytic solutions for the two-level systems[15,16]:

(1) *Carroll's and Hioe's Result*[15]: *Class I.* Here we have a class of amplitude and detuning functions of the forms

$$\Omega_0 = \Omega_0{}^* = \frac{\alpha}{\pi}\frac{1}{[z(1-z)]^{1/2}}\frac{dz}{dt},$$

$$\Delta_0 = \frac{1}{\pi}\left(\frac{\beta}{z} + \frac{\gamma}{1-z}\right)\frac{dz}{dt}, \tag{24}$$

where $z(t)$, which goes along the real axis from 0 to 1 as t increases from $-\infty$ to $+\infty$, is an arbitrary function of time that does not affect the final level occupation probabilities. For example, a hyperbolic secant or a Lorentzian pulse shape for $\Omega_0(t)$ can be obtained[14,15] by setting $z = \frac{1}{2}[1 + \tanh(t)]$ or $z = \frac{1}{2}[1 + t/(t^2 + 1^2)^{1/2}]$. Other examples are given in Ref. 15. The final level occupation probabilities $|a(t)|^2$ and $|b(t)|^2$ at $t = +\infty$ depend on the parameters α, β, and γ and are given by

$$|a(+\infty)|^2 = [\cosh(\beta + \gamma) + \cos\Phi]/[2\cosh(\beta)\cosh(\gamma)]$$

and

$$|b(+\infty)|^2 = 1 - |a(+\infty)|^2, \tag{25}$$

where

$$\Phi = [\alpha^2 - (\beta - \gamma)^2]^{1/2}. \tag{26}$$

In particular, complete return of population requires that $\beta = \gamma$, $\Phi = 2n\pi$, $n = 1, 2, \ldots$, and complete transfer of population requires that $\beta = -\gamma$, $\Phi = (2n - 1)\pi$.

(2) *Carroll's and Hioe's Results*[16]: *Class II.* Here we have another class of amplitude and detuning functions of the forms

$$\Omega_0 = \Omega_0{}^* = \frac{\alpha}{\pi}\frac{1}{z^2+1}\frac{dz}{dt},$$

$$\Delta_0 = \frac{1}{\pi}\frac{-(\beta-\gamma)z + (\beta+\gamma)}{z^2+1}\frac{dz}{dt}, \tag{27}$$

where $z(t)$, which goes along the real axis from $-\infty$ to $+\infty$ as t increases from $-\infty$ to $+\infty$, is again an arbitrary function of time that does not affect the final level occupation probabilities. A hyperbolic secant or a Lorentzian pulse shape for $\Omega_0(t)$ can be obtained[14,15] by setting $z = \sinh(t)$ or $z = t$. The final level occupation probabilities are given by $|a(+\infty)|^2 = [2\cosh(\beta - \gamma)\cos(r)\cos(s) - \cos^2(r) - \cos^2(s)]/\sinh^2(\beta - \gamma)$ and

$$|b(+\infty)|^2 = 1 - |a(+\infty)|^2, \tag{28}$$

where

$$r = \tfrac{1}{2}\{\alpha^2 + [\beta + \gamma + i(\beta - \gamma)]^2\}^{1/2}, \tag{29a}$$

$$s = \tfrac{1}{2}\{\alpha^2 + [\beta + \gamma - i(\beta - \gamma)]^2\}^{1/2}. \tag{29b}$$

In particular, the case $\beta = \gamma$ gives the special case when $\Omega_0(t)$ and $\Delta_0(t)$ have the same time dependence; as it turns out, the case $\beta = -\gamma$ gives the same result as the case $\beta = -\gamma$ for class I.

Substitutions of Eqs. (25) and (28) into Table 1 give us the solutions of the N-level SU(2) model that have not been presented previously.

(3) *The Generalized Landau–Zener Result for N-Level Systems.* Here we have

$$\Omega_0 = \text{const.}, \qquad \Delta_0(t) = r_0 t, \tag{30}$$

where r_0 = constant. As t ranges from $-\infty$ to $+\infty$, Landau[17] and Zener[18] gave

$$|a(+\infty)|^2 = P, \qquad |b(+\infty)|^2 = 1 - P, \tag{31}$$

where

$$P = \exp(-\pi|\Omega_0|^2/2r_0), \tag{32}$$

assuming that the initial occupation probabilities of levels 1 and 2 at $t = -\infty$ are 1 and 0, respectively.

It was also known that

$$|a(0)|^2 = \tfrac{1}{2}(1 + P^{1/2}), \qquad |b(0)|^2 = \tfrac{1}{2}(1 - P^{1/2}). \tag{33}$$

It can be verified that, using Table 1 and Eqs. (31) and (33), we can immediately write down the three-level ($N = 3$) generalization of Landau–Zener result given in Ref. 19 without having to perform the somewhat complicated analysis given in that reference. Substitutions of Eqs. (31) and (33) into Table 1 or Eq. (12) give us the N-level generalization of the Landau–Zener result.

In conclusion, we have derived a simple analytic solution [Eq. (12)] for what we called the SU(2) model in which the time-dependent Hamiltonian of the system satisfies Eqs. (7). The system permits two independent arbitrary functions of time, $\Delta_0(t)$ and $\Omega_0(t)$, and the solution expresses the probability amplitudes $C_m{}^{(j)}(t)$ of the N-level system in terms of the fundamental solution $a(t)$ and $b(t)$ of Eq. (8). This enables us to apply many recently derived analytic results for the two-level system to the N-level systems belonging to the SU(2) model. In particular, the conditions for complete transfer of population from level 1 to level N, for complete return of population, and for complete depletion of population of a particular level are given in simple expressions.

If the Hamiltonian $\hat{H}'(t)$ of an N-level system in the time-dependent Schrödinger equation

$$i\frac{d}{dt}\mathbf{C}'(t) = \hat{H}'(t)\mathbf{C}'(t)$$

is not of a tridiagonal form but can be transformed by a time-independent unitary matrix $\hat{U}$ such that the transformed Hamiltonian $\hat{H}(t) = \hat{U}^\dagger\hat{H}'(t)\hat{U}$ is tridiagonal and its elements satisfy Eqs. (7), then clearly our solution for $\mathbf{C}(t)$ corresponding to the Hamiltonian $\hat{H}(t)$ again applies, from which we can deduce the probability amplitudes $\mathbf{C}'(t)$ corresponding to the original Hamiltonian $\hat{H}'(t)$ to be

$$\mathbf{C}'(t) = \hat{U}\mathbf{C}(t).$$

Finally, we should mention that our solution can also be applied to the mixed-state problem formulated in terms of the density matrix. By using the Racah tensors, the N^2-dimensional dynamical space can be decomposed,[5,6] when the Hamiltonian of the system satisfies the SU(2) symmetry, into N independent subspaces of dimensions 1, 3, 5, ..., $2N - 1$, respectively, with the equations of motion in each of these subspaces having the forms[6] given by Eqs. (1) and (7). More results in this connection will be published elsewhere.

ACKNOWLEDGMENT

This research is partially supported by the Department of Energy, Office of Basic Energy Sciences, Division of Chemical Sciences, under grant no. DE-FG02-84-ER13243.

REFERENCES AND NOTES

1. R. J. Cook and B. W. Shore, Phys. Rev. A **20**, 539 (1979).
2. F. T. Hioe, Phys. Rev. A **26**, 1466 (1982).
3. F. T. Hioe, Phys. Rev. A **28**, 879 (1983).
4. F. T. Hioe and J. H. Eberly, Phys. Rev. A **29**, 1164 (1984).
5. F. T. Hioe, Phys. Rev. A **30**, 3097 (1984).
6. F. T. Hioe, "Analytic solutions and dynamic symmetries in laser driven atomic excitations," in *Advances in Chemical Physics*, J. O. Hirschfelder, R. E. Wyatt, and R. D. Coalson, eds. (Wiley, New York, to be published).
7. See, e.g., L. I. Schiff, *Quantum Mechanics*, 2nd ed. (McGraw-Hill, New York, 1955), p. 145.
8. See, e.g., M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley, Reading, Mass., 1962), p. 354.
9. See, e.g., A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. Press, Princeton, N.J., 1957), p. 55.
10. M. Appell, Compt. Rend. **XCI**, 28 (1880).
11. See, e.g., A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, eds. *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1, p. 151 and Vol. 2, p. 158.
12. F. T. Hioe, Phys. Lett. **99A**, 150 (1983).
13. J. Oreg, F. T. Hioe, and J. H. Eberly, Phys. Rev. A **29**, 690 (1984).
14. The equations below Eq. (16) in Ref. 12 should be corrected to read as

$$d_0(t) = (-1)^{N-1} D(t)^{-1} \prod_{j=2}^{N} \lambda_j(t), \ldots,$$

$$d_{N-1}(t) = D(t)^{-1}$$

$$D(t) = \prod_{j=2}^{N} [\lambda_1(t) - \lambda_j(t)].$$

15. F. T. Hioe and C. E. Carroll, Phys. Rev. A **32**, 1541 (1985).
16. C. E. Carroll and F. T. Hioe, J. Phys. A **19**, 3579 (1986).
17. L. Landau, Phys. Z. Sowjetunion **2**, 46 (1932).
18. C. Zener, Proc. R. Soc. London Ser. A **137**, 696 (1932).
19. C. E. Carroll and F. T. Hioe, J. Phys. A **19**, 1151 (1986); see also C. E. Carroll and F. T. Hioe, J. Phys. A **19**, 2061 (1986) for the three-level Landau–Zener model that does not belong to our SU(2) model.